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LOCAL STRUCTURE OF REPRESENTATION VARIETIES

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ABSTRACT

The isomorphism classes of simple representations of a finitely generated group form an algebraic variety. It is shown that the tangents to formal curves through a point in the variety are given by the linear parts of Lie algebra homomorphisms from the Lie algebra of the pro-unipotent radical of the universal pro-affine hull of the group to the Lie algebra of matrices over formal power series. This description allows a determination of singular representations for nilpotent groups and of the tangent cones to representation varieties for abelian-by-finite groups.

The isomorphism classes of simple n-dimensional representations of the finitely generated group Γ form an algebraic variety: its points are conjugacy classes of tuples of $n \times n$ matrices (the images of the generators) which satisfy the relations of the group (so give a representation) and which linearly span all matrices (so give a simple representation). We denote this variety $S_n(\Gamma)$. It is a comparatively straightforward observation that the tangent space to an isomorphism class $[\rho]$ is contained in the cohomology space $H^1(\Gamma, \mathrm{Ad} \circ \rho)$, where Ad is the adjoint representation, so for γ in Γ and A an $n \times n$ matrix $(\mathrm{Ad} \circ \rho)(\gamma)(A) = \rho(\gamma)A\rho(\gamma)^{-1}$; see [8], [4, 2.2, p. 33], or Section 2 below. It is a much more delicate matter to locate the tangent space at $[\rho]$ inside of H^1. (It can be a proper subspace, as shown in [4, 2.10, p. 42] and Example 4.6 below.) The (reduced) tangent cone at $[\rho]$ is equally mysterious. However, as the main results of this paper show, it is possible to give a good description of a subset of the tangent cone and space which we term the space of curve tangents, namely the linear tangents to formal curves in $S_n(\Gamma)$ through $[\rho]$. Since at non-singular points the tangent space and the space of curve tangents coincide, this provides a generic description of tangent spaces, as well as a necessary condition for non-singularity (linearity of the space of curve tangents). And in the case of

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abelian-by-finite groups, where it is shown (4.1) that the spaces of curve tangents and tangent cones always coincide, we have a description of the latter and have a determination of the dimension of the representative variety.

To understand the description of the space of curve tangents, we need first to recall the universal pro-affine hull of Γ. This proalgebraic group, denoted $A(\Gamma)$, receives a homomorphism from Γ with Zariski-dense image through which every representation of Γ factors. The prounipotent radical $U(\Gamma)$ of $A(\Gamma)$ is normalized by Γ and hence its Lie algebra is a Γ-module. For our representation ρ of degree n, the curve tangents to $S_n(\Gamma)$ at $[\rho]$ correspond to those Γ-equivariant linear maps from Lie($U(\Gamma)$) made abelian to the space of $n \times n$ matrices which can be extended to Γ-equivariant Lie algebra homomorphisms from Lie($U(\Gamma)$) to the Lie algebra of $n \times n$ matrices over formal power series without constant term.

The above description, while not in general effective, has the following nice form when Lie($U(\Gamma)$) is finite-dimensional and graded by its lower central series: then the extentable linear maps are just the Lie homomorphisms. This allows the result to be applied to nilpotent and abelian-by-finite groups.

The paper is organized as follows: Part 1, Preliminaries, reviews definitions and establishes notations for representation varieties and their tangent objects. Part 2, Tangent spaces, proves the main results describing spaces of curve tangents in terms of linear maps extendable to Lie homomorphisms. Part 3, Nilpotent groups, determines that the non-singular semi-simple representations in the nilpotent case are just those without repeated composition factors. Part 4, Abelian-by-finite groups, applies the theory to this case, taking into account the structure of representation varieties implied by the Clifford theory.

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1. Preliminaries

We use the following notations and conventions from [4] throughout:

k denotes an algebraically closed field of characteristic zero (usually $\mathbf{C}$).

$\Gamma = \langle g_1, \ldots, g_d \mid r_q, q \in Q \rangle$ denotes a finitely generated group with generators g_p and relations r_q.

G denotes an affine algebraic group scheme over k.

The functor from commutative k-algebras to sets which sends the algebra A to Hom(Γ, $G(A)$) is represented by an affine scheme $\mathcal{R}(\Gamma, G)$ of finite type over

k and whose k-points comprise an affine algebraic set denoted $R(\Gamma, G)$. When $G = \mathrm{GL}_n$ we use the notation $\mathscr{R}_n(\Gamma)$ and $R_n(\Gamma)$. The coordinate rings of $\mathscr{R}_n(\Gamma)$ and $R_n(\Gamma)$ are denoted $\mathscr{A}_n(\Gamma)$ and $A_n(\Gamma)$. (These are equal only if $\mathscr{R}_n(\Gamma)$ is reduced.)

The group scheme G acts on $\mathscr{R}(\Gamma, G)$ by conjugation, and when G is reductive a universal categorical quotient exists. This action and its quotient are most important for the case $G = \mathrm{GL}_n$, where the categorical quotient is denoted $\mathscr{S}\mathscr{S}_n(\Gamma)$. Inside $\mathscr{R}_n(\Gamma)$ is the GL_n-stable open subscheme $\mathscr{R}_n(\Gamma)^s$ of simple representations (ρ in $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(A))$ is simple if $\rho(\Gamma)$ spans $M_n(A)$ over A). GL_n acts with closed orbits on $\mathscr{R}_n(\Gamma)^s$ and its image $\mathscr{S}_n(\Gamma)$ in $\mathscr{S}\mathscr{S}_n(\Gamma)$ is a geometric quotient of $\mathscr{R}_n(\Gamma)^s$, locally trivial for the étale topology. The k points of these schemes are denoted $SS_n(\Gamma)$ and $S_n(\Gamma)$.

We need to briefly review the hierarchy of tangential structures to schemes at k-points. We also use this notation throughout.

For the purposes of this review, we let $\mathscr{R}$ denote an affine k-scheme of finite type with coordinate ring $\mathscr{A}$ and variety $R = \mathscr{R}(k)$ of k points. $A = \mathscr{A}_{\mathrm{red}}$ is the coordinate ring of R. We fix a k-point r in R and let $\mathfrak{m} \subseteq A$ be the corresponding maximal ideal. In the following definitions $k[\varepsilon]$ denotes the dual numbers over k:

The *tangent space* $T_r(\mathscr{R})$ is the fibre of $\mathscr{R}(k[\varepsilon]) \to \mathscr{R}(k) = R$ over r.
The *tangent space* $T_r(R)$ is the fibre of $\mathrm{Alg}_k(A, k[\varepsilon]) \to \mathrm{Alg}_k(A, k) = R$ over r.
The *tangent cone* $TC_r(R)$ is $\mathrm{Alg}_k(\mathrm{gr}_{\mathfrak{m}}(A), k)$.

We also need to consider formal curves in $\mathscr{R}$ at r and their tangents.

Definition 1.1. The set $C_r(\mathscr{R})$ of *formal curves* in $\mathscr{R}$ at r is the fibre of $\mathscr{R}(k[[t]]) \to \mathscr{R}(k) = R$ over r. The image of $C_r(\mathscr{R})$ in $T_r(R)$ under the natural map $k[[t]] \to k[\varepsilon]$ is denoted $CT_r(R)$ and called the set of *curve tangents* to R at r.

For general schemes one needs also to consider higher-order curve tangents. For this, we define $k_m[[t]]$ to the subring of $k[[t]]$ of all series $\Sigma\, a_i t^i$ with $a_i = 0$ for $1 < i < m$. Let $C_r(\mathscr{R})^m$ be the set of formal curves f with $f(\mathscr{A})$ in $k_m[[t]]$ and let $CT_r(R)^m$ denote the image of $C_r(\mathscr{R})^m$ in $T_r(R)$ under the map $k_m[[t]] \to k[\varepsilon]$ defined by the inclusion into $k[[t]]$ followed by reduction modulo t^{m+1}. Note that $k_1[[t]] = k[[t]]$ and $CT_r(R)^1 = CT_r(R)$.

These sets are related as follows:

$$CT_r(R) \subseteq CT_r(R)^m \subseteq TC_r(R) \subseteq T_r(R) \subseteq T_r(\mathscr{R}), \tag{1.2a}$$

$$TC_r(R) = \bigcup\{CT_r(R)^m \mid m \geqq 1\}. \tag{1.2b}$$

The inclusions of (1.2a) arise as follows: if the curve tangent τ comes from the curve $f: \mathscr{A} \to k[[t]]$ it also comes from f composed with $k[[t]] \to k[[t^m]]$ by $t \to t^m$, where the latter is regarded as a subring of $k_m[[t]]$, and hence the first inclusion. If the curve tangent τ comes from the curve $f: \mathscr{A} \to k_m[[t]]$, then f factors through A and $f(a) = r_0(a) + r_1(a)t^m +$ higher degree, where $r_0 = r$ and $r_0 + r_1\varepsilon = \tau$. Let M_i denote the ideal $\mathfrak{m}^{m-1+i} \cap k_m[[t]]$. Then $f(\mathfrak{m}^i) \subseteq M_i$ and f induces a map of associated gradeds $\mathrm{gr}(A) \to \mathrm{gr}(k_m[[t]]) = k[t]$. Following this by evaluation at 1 gives a homomorphism $h: \mathrm{gr}(A) \to k$ in the tangent cone. This is the second inclusion. For any homomorphism $g = \oplus g_i$ in $TC_r(R)$, the corresponding tangent vector is σ defined by

$$\sigma(a) = r(a) + g_1((a - r(a)) + m^2)\varepsilon.$$

This is the third inclusion of (1.2a), and the fourth is clear. Equality (1.2b) is standard.

The inclusions of (1.2a) can all be proper. By definition r is *non-singular* on R when $TC_r(R) = T_r(R)$. When this happens, $CT_r(R) = T_r(R)$ also [6, Thm. 61, p. 206], so we can also take the latter equality as the definition. If also $\mathscr{R}$ is reduced at r then $CT_r(R) = T_r(\mathscr{R})$ and in this case we say R is *scheme non-singular* at r.

We will want to specialize these definitions to the case $\mathscr{R} = \mathscr{R}_n(\Gamma)$ and $r = \rho \in R_n(\Gamma)$. Thus we need to consider

$$\mathscr{R}_n(\Gamma)(k[\varepsilon]) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k[\varepsilon]))$$

and

$$R_n(\Gamma)(k[[t]]) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k[[t]])).$$

Both of these are sets of homomorphism from Γ to (pro)-algebraic groups. We recall that such homomorphisms are specializations of a universal one [4, 4.2, p. 65]: There is a pro-affine algebraic group $A(\Gamma)$ and a group homomorphism $j: \Gamma \to A(\Gamma)$ with Zariski-dense image such that any homomorphism $h: \Gamma \to G$ from Γ to a pro-algebraic group G factors as $h = \bar{h}j$ where $\bar{h}: A(\Gamma) \to G$ is a pro-algebraic group homomorphism. The pro-algebraic group $A(\Gamma)$ is a semi-direct product of its prounipotent radical $U(\Gamma)$ and any maximal proreductive subgroup [2]. These latter are all conjugate by elements of $U(\Gamma)$ and we fix one and denote it $P(\Gamma)$. $U(\Gamma)$ is acted on by Γ via j.

2. Tangent spaces

Throughout this section we fix a finitely generated group Γ and a *semi-simple* representation $\rho: \Gamma \to \mathrm{GL}_n(k)$. In [4, 2.2, p. 33] it was shown that $T_\rho(\mathscr{R}_n(\Gamma))$ can be identified with $Z^1(\Gamma, \mathrm{Ad} \circ \rho)$. (This does not require ρ to be semi-simple.) The identification was based on writing $\mathrm{GL}_n(A)(k[\varepsilon])$ as the semi-direct product $M_n(k) \rtimes \mathrm{GL}_n(A)(k)$ using the adjoint action of $\mathrm{GL}_n(A)$ on M_n.

We will need an alternate description, based on writing $\mathrm{GL}_n(A)(k[\varepsilon])$ as a semi-direct product of its subgroups $I + \varepsilon M_n(k)$ and $\mathrm{GL}_n(A)(k)$, which makes the role of $U(\Gamma)$ more explicit. This allows us to describe the set $CT_\rho(R_n(\Gamma))$ in terms of $U(\Gamma)$. We begin with a calculation of coboundaries:

LEMMA 2.1. *Let $\rho_\varepsilon : \Gamma \to \mathrm{GL}_n(k[\varepsilon])$ in $T_\rho(\mathscr{R}_n(\Gamma)) = T_\rho$ correspond to σ in $Z^1(\Gamma, \mathrm{Ad} \circ \rho) = Z^1$ and let $A = I + \varepsilon D$ where $D \in M_n(k) = C^0(\Gamma, \mathrm{Ad} \circ \rho)$. Then $A\rho_\varepsilon A^{-1}$ in T_ρ corresponds to $\sigma + \delta(D)$ in Z^1.*

PROOF. We identify $B + C\varepsilon$ in $\mathrm{GL}_n(k[\varepsilon])$ with (CB^{-1}, B) in $M_n(k) \rtimes \mathrm{GL}_n(k)$ to relate T_ρ and Z^1 [4, 2.1, p. 32]. If $\rho_\varepsilon(a) = B_a + \varepsilon C_a$, then $\sigma(a) = C_a B_a^{-1}$. Since $A^{-1} = I - \varepsilon D$, $A\rho_\varepsilon A^{-1}(a) = B_a + \varepsilon(C_a + DB_a - B_a D)$ which corresponds to the cocycle sending a to $C_a + B_a - B_a D B_a^{-1}$; i.e., to $\sigma + \delta(D)$.

The semi-simple representation ρ extends to a homomorphism $\bar{\rho} : A(\Gamma) \to \mathrm{GL}_n(k)$. If $\phi : U(\Gamma) \to I + \varepsilon M_n(k)$ is a homomorphism, we can define a function $\phi\bar{\rho} : A(\Gamma) = U(\Gamma)P(\Gamma) \to \mathrm{GL}_n(k[\varepsilon])$ by $(\phi\bar{\rho})(up) = \phi(u)\bar{\rho}(p)$. For $\phi\bar{\rho}$ to be a homomorphism, it is necessary and sufficient that

$$\phi(pup^{-1}) = \bar{\rho}(p)\phi(u)\bar{\rho}(p)^{-1} \qquad \text{for all } u \in U(\Gamma) \text{ and } p \in P(\Gamma).$$

This conjugation formula is implied by the more general formula

$$\phi(aua^{-1}) = \bar{\rho}(a)\phi(u)\bar{\rho}(a)^{-1} \qquad \text{for } u \in U(\Gamma) \text{ and } a \in A(\Gamma),$$

and is equivalent to it when $\bar{\rho}(U(\Gamma))$ commutes with $I + \varepsilon M_n(k)$. Since $\bar{\rho}(U(\Gamma))$ is always in $\mathrm{GL}_n(k)$, we get commutativity if $\bar{\rho}(U(\Gamma)) = I$, which happens when ρ is semi-simple. Thus in the semi-simple case $\phi\bar{\rho}$ is a homomorphism if and only if ϕ commutes with the Γ, hence $A(\Gamma)$, action on $U(\Gamma)$ and $I + \varepsilon M_n(k)$. Thus we have map

$$\Phi : \mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k)) \to \mathrm{Hom}(A(\Gamma), \mathrm{GL}_n(k[\varepsilon])) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k[\varepsilon])).$$

Since composing $\Phi(\phi) = \phi\bar{\rho}$ with the map $\mathrm{GL}_n(k[\varepsilon]) \to \mathrm{GL}_n(k)$ by $\varepsilon \to 0$ gives ρ, the image of Φ lies in $T_\rho(\mathscr{R}_n(\Gamma))$..In fact, the image is most of the tangent space.

THEOREM 2.2. *Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be a semi-simple representation, then*

$$T_\rho(\mathscr{R}_n(\Gamma)) = \mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k)) \oplus B^1(\Gamma, \mathrm{Ad} \circ \rho).$$

PROOF. The map Φ is injective since $\phi\bar{\rho} \mid U(\Gamma) = \phi$. Suppose $D \in M_n(k)$ and let $\delta(D) \in B^1(\Gamma, \mathrm{Ad} \circ \rho)$ be the corresponding co-boundary. Assume $\delta(D) = \phi\bar{\rho}$ for some ϕ, say

$$\phi(u) = I + \varepsilon C_u \quad \text{for } u \in U(\Gamma) \qquad \text{and} \qquad \bar{\rho}(p) = B_p \quad \text{for } p \in P(\Gamma).$$

Then $\phi\bar{\rho}(up) = E_{up} + \varepsilon F_{up}$ where $E_{up} = B_p$ and $F_{up} = C_u B_p$. The cocycle corresponding to $\phi\bar{\rho}$ is then $up \to F_{up}E_{up}^{-1} = C_u$. Now

$$\delta(D)(up) = D - \bar{\rho}(up)D\bar{\rho}(up)^{-1} = D - B_p D B_p^{-1}.$$

Thus $C_u = D - B_p D B_p^{-1}$ for all u, p. Taking $p = e$ we see that $C_u = 0$ for all u, so ϕ is trivial. This shows that the sum is direct. To see that it is all of T_ρ, let $\rho_\varepsilon : \Gamma \to \mathrm{GL}_n(k[\varepsilon])$ be a homomorphism over ρ. Then $\bar{\rho}_\varepsilon(P(\Gamma))$ is reductive, so there is an A in $I + \varepsilon M_n(k)$ with $A^{-1}\bar{\rho}_\varepsilon A = f$ sending $P(\Gamma)$ to $\mathrm{GL}_n(k)$: for $I + \varepsilon M_n(k)$ is the unipotent radical and $\mathrm{GL}_n(k)$ a maximal reductive subgroup of $\mathrm{GL}_n(k[\varepsilon])$. Under evaluation of ε at 0, f and $\bar{\rho}_\varepsilon$ have the same image $\bar{\rho}$, so $f(U(\Gamma)) \subseteq I + \varepsilon M_n(k)$. Let $\phi = f \mid U(\Gamma)$. Then $f = \phi\bar{\rho}$: for $f(up) = f(u)f(p) = \phi(u)f(p)$, and since $f(p) \in \mathrm{GL}_n(k)$ we can determine its value by setting ε to 0, so $f(p) = \bar{\rho}(p)$. Since f is a homomorphism, we have ϕ in $\mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k))$. If $A = I + \varepsilon D$, then by (2.1) we have $\rho_\varepsilon = \phi + \delta(D)$, completing the proof of (2.2).

In fact, (2.2) was essentially obtained by other means in [4, 4.7, p. 73]. We have provided the new proof to facilitate the following result, which describes the space of curve tangents.

THEOREM 2.3. *Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be a semi-simple representation, let*

$$V = \{\phi \in \mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k)) \mid \exists\, \phi_t \in \mathrm{Hom}_\Gamma(U(\Gamma), I + tM_n(k[[t]])) \text{ with } \phi_t \equiv \phi \pmod{t^2}\}.$$

Then

$$CT_\rho(R_n(\Gamma)) = V \times B^1(\Gamma, \mathrm{Ad} \circ \rho).$$

If ρ is simple, then

$$CT_{[\rho]}(S_n(\Gamma)) = V.$$

PROOF. The group $\mathrm{GL}_n(k[[t]])$ is pro-algebraic and the semi-direct product of its pro-unipotent radical $I + tM_n(k[[t]])$ and a maximal reductive subgroup $\mathrm{GL}_n(k)$. As above, a homomorphism ψ_t in $\mathrm{Hom}_\Gamma(U(\Gamma), I + tM_n(k[[t]]))$ yields a homomorphism $\psi_t\bar{\rho} : A(\Gamma) \to \mathrm{GL}_n(k[[t]])$, which is a formal curve at ρ. The corresponding curve tangent ψ is then obtained by reading ψ_t modulo t^2. If $\psi_t\bar{\rho}$ is composed with inner automorphism by

$$A = I + tD + t^2 D_2 + \cdots + t^s D_s + \cdots$$

the resulting curve tangent is conjugated by $I + \varepsilon D$ and hence, by (2.1), is

$\psi + \delta(D)$. Thus $V \times B^1$ consists of curve tangents. If a curve tangent at ρ comes from the curve ρ_t, then $\bar{\rho}_t(P(\Gamma))$ is reductive so we can conjugate by an A in $I + tM_n(k[[t]])$ to get $A^{-1}\bar{\rho}_t A = g$ sending $P(\Gamma)$ to $\mathrm{GL}_n(k)$. As in (2.2), this implies that $g = \psi_t \bar{\rho}$ for suitable ψ_t in $\mathrm{Hom}_\Gamma(U(\Gamma), I + tM_n(k[[t]]))$. Thus the curve tangent given by ρ_t is ψ_t modulo t^2 plus a coboundary coming from A, as in (2.2). This shows that $CT_\rho = V \times B^1$.

For the assertion about the simple case, we use that $p : R_n(\Gamma)^s \to S_n(\Gamma)$ is locally trivial for the étale topology. Let G denote $\mathrm{PGL}_n(k)$, let X be an affine G-stable neighborhood of ρ and let Y denote the quotient X/G. There is a subvariety W of X through ρ such that $G \times W \to X$ is étale with open image and $W \to Y$ is étale (W is an étale slice in the sense of Luna). Since étale morphisms admit unique liftings through square zero extensions, it is easy to see that

$$C_{(e,\rho)}(G \times W) = C_e(G) \times C_\rho(W) = C_\rho(X)$$

so that $CT_\rho(X) = T_e(G) \times CT_\rho(W)$ and that $CT_\rho(W) = CT_{p(\rho)}(Y)$. Since the image of $T_e(G)$ in $T_\rho(X)$ is $B^1(\Gamma, \mathrm{Ad} \circ \rho)$ [4, 2.3, p. 34], this implies that $CT_{[\rho]}(S_n(\Gamma)) = V$.

As we shall see, (2.3) is most useful in tests for singularity, since what it does is provide a concrete description of the set of curve tangents inside the space of scheme tangents. We will thus, for example, be able to test for non-linearity of the set of curve tangents (which will imply non-singularity). Since the tangent space is only embedded in the (cocycle and cohomology) spaces of scheme tangents, (2.3) can't be used to assess singularity in general. However, we can use it to assess "scheme non-singularity", a remark which we note for future reference as a corollary to (2.3).

COROLLARY 2.4. *Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be a semi-simple representation, and let V be as in* (2.3). *Then if $V = \mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k))$, ρ is non-singular on $\mathscr{R}_n(\Gamma)$ and*

$$T_\rho(R_n(\Gamma)) = T_\rho(\mathscr{R}_n(\Gamma)) = Z^1(\Gamma, \mathrm{Ad} \circ \rho).$$

If in addition ρ is simple, then ρ is non-singular on $S_n(\Gamma)$ and $T_\rho(S_n(\Gamma)) = H^1(\Gamma, \mathrm{Ad} \circ \rho)$.

PROOF. The hypothesis, by (2.3), is equivalent to the statement that $CT_\rho(R_n(\Gamma)) = T_\rho(\mathscr{R}_n(\Gamma))$ from which the conclusions all follow.

The essential point of (2.3) is that up to conjugation (whose tangential action is by addition of coboundaries) formal curves in $R_n(\Gamma)$ at ρ correspond to Γ-equivarient homomorphisms from $U(\Gamma)$ to $I + tM_n(k[[t]])$. Both of these are pro-unipotent groups, and homomorphisms between them are determined by the corresponding maps on Lie algebras [3, 1.1, p. 78], these being here $\mathrm{Lie}(U(\Gamma))$ and $tM_n(k[[t]])$.

Thus to use (2.3) effectively, we need to understand the function

$$\mathrm{Hom}_\Gamma(U(\Gamma), I + tM_n(k[[t]])) \to \mathrm{Hom}_\Gamma(U(\Gamma), I + \varepsilon M_n(k)) \tag{2.5}$$

and its Lie algebra form

$$\mathrm{Lie}_\Gamma(L, tM_n(k[[t]])) \to \mathrm{Lie}_\Gamma(L, \varepsilon M_n(k)), \tag{2.6}$$

where $L = \mathrm{Lie}(U(\Gamma))$.

In fact, (2.6) makes sense for any (pro) nilpotent Lie algebra L, and we will look at it in that generality. It is especially important in (2.6) to note that $\varepsilon M_n(k)$ is an *abelian* Lie algebra, and it is not $M_n(k)$ under bracket conjugation.

To study (2.6), we will temporarily drop the Γ-action. A Lie homomorphism $\phi : L \to tM_n(k[[t]])$ induces homomorphisms on the terms of the lower central series and hence a homomorphism of associated graded Lie algebras. (We recall that $\mathscr{C}^1L = L$ and $\mathscr{C}^{i+1}L = [L, \mathscr{C}^iL]$, that $\mathrm{gr}_i(L) = \mathscr{C}^iL/\mathscr{C}^{i+1}L$ and that $\mathrm{gr}(L) = \bigoplus \mathrm{gr}_i(L)$.) Let M denote $tM_n(k[[t]])$. Then a simple induction shows that $\phi(\mathscr{C}^iL) \subseteq t^iM$, the latter being an ideal of M. Since $[t^iM, t^jM] \subseteq t^{i+j}M$, $\bigoplus(t^iM/t^{i+1}M)$ is a graded Lie algebra canonically isomorphic to $tM_n(k[t])$. From ϕ there is an induced homomorphism $\mathrm{gr}(\phi) : \mathrm{gr}(L) \to M$. This construction has implications for (2.6), as we show in (2.7). For simplicity, we will restrict to finite-dimensional L. In general the argument is similar, except for non-finitely generated L the closed lower central series must be used.

Lemma 2.7. *Let L be a finite-dimensional nilpotent Lie algebra with minimal generating set $x_1, \ldots, x_d$. Let $\phi : L \to tM_n(k[[t]])$ be a Lie homomorphism, and suppose $\phi(x_i) = tA_i$ + higher degree. Then the Lie subalgebra of $\mathscr{GL}_n(k)$ generated by $A_1, \ldots, A_d$ is a homomorphic image of* $\mathrm{gr}(L)$.

Proof. We replace ϕ, L, $tM_n(k[[t]])$ by $\mathrm{gr}(\phi)$, $\mathrm{gr}(L)$, $tM_n(k[t])$. So we assume $L = L^1 \oplus \cdots \oplus L^s$ is graded and L^i has a basis $x_1^i, x_2^i, \ldots$. Let $\phi(x_p^i) = t^iA_i(p)$. Since $[x_p^i, x_q^j] = \sum \ell_{i,p,j,q,r} x_r^{i+j}$, we have, comparing lowest degree terms, that $t^{i+j}[A_i(p), A_j(q)] = \sum \ell_{i,p,j,q,r} t^{i+j} A_{i+j}(r)$. Cancel t^{i+j} from both sides. The resulting equality implies that $x_p^i \to A_i(p)$ extends linearly to a representation of L in $\mathscr{GL}_n(k)$. Since L is generated by L^1, this means that $A_1(1), A_1(2), \ldots$ generate a homomorphic image of L. In terms of our original (ungraded) L, the minimal generators x_i go to a basis of L^1. So we can assume that $x_i^1 = x_i + \mathscr{C}^2L$, and the result follows.

The graded case, used to prove (2.7), is worthwhile noting separately. In fact, in this case we also have a converse:

PROPOSITION 2.8. *Let L be a finite-dimensional graded nilpotent Lie algebra with minimal generating set $x_1, \ldots, x_d$. If $\phi: L \to tM_n(k[[t]])$ is a Lie homomorphism with $\phi(x_i) = tA_i$ + higher degree, then the Lie subalgebra of $\mathcal{G}\ell_n(k)$ generated by $A_1, \ldots, A_d$ is a homomorphic image of L. Conversely, if L is represented in $\mathcal{G}\ell_n(k)$ with x_i represented by B_i then there is a Lie homomorphism $\psi: L \to tM_n(k[[t]])$ with $\psi(x_i) = tB_i$.*

PROOF. We need only show the converse, and we adopt the notation of the proof of (2.7). Then x_p^i is represented in $\mathcal{G}\ell_n(k)$ by $B_i(p)$, where $B_1(i) = B_i$. Then it is clear that $\psi: L \to tM_n(k[[t]])$ defined by $\psi(x_p^i) = t^iB_i(p)$ and linearity is a Lie homomorphism.

Proposition 2.8 basically describes the image of (2.6), ignoring the Γ-action, in the graded case. We do not know if (2.8) holds for all nilpotent Lie algebras. As we see below, (2.8) suffices in two special cases: (i) Γ nilpotent and (ii) $U(\Gamma)$ abelian. The technique used to prove (2.7) and (2.8) also has important implications for the tangent cones in those cases.

THEOREM 2.9. *Assume $L = \mathrm{Lie}(U(\Gamma))$ is graded by its lower central series. Let $\rho: \Gamma \to \mathrm{GL}_n(k)$ be semi-simple. Then $CT_\rho(R_n(\Gamma)) = TC_\rho(R_n(\Gamma))$. If ρ is simple, then $CT_{[\rho]}(S_n(\Gamma)) = TC_{[\rho]}(S_n(\Gamma))$.*

PROOF. By (1.26), it will suffice to show that $CT_\rho(R_n(\Gamma))^m = CT_\rho(R_n(\Gamma))$ for all m. Now the essential point of the proof of (2.3) is that, up to conjugation by elements of $I + tM_n(k[[t]])$, formal curves in $R_n(\Gamma)$ at ρ correspond to Lie homomorphisms. Those higher-order curves of order m correspond to Lie homomorphisms $\phi: L \to t^mM_n(k[[t]])$. We use the notation $L = L^1 \oplus \cdots \oplus L^s$, where L^i has basis $x_1^i, x_2^i, \ldots$ as in the proof of (2.6). Suppose that $[x_p^i, x_q^j] = \Sigma \ell_{i,j,p,r} x_r^{i+j}$ and that $\phi(x_p^i) = t^{mi}A_i(p)$ + higher degree. Exactly as in (2.7), it follows that $X_p^i \to A_i(p)$ extends linearly to a representation of L. This representation is, on the one hand, the order-m curve tangent to ϕ at ρ, and on the other, by (2.8), is an order-one curve tangent at ρ also. This proves that $CT_\rho^m = CT_\rho$ and hence the theorem.

It's possible that (2.9) holds in more, even complete, generality. We will apply it only in the case that L is abelian, hence (trivially) graded.

3. Nilpotent groups

Throughout this section Γ will denote a finitely generated nilpotent group. In this case $U(\Gamma)$ is a (finite-dimensional) unipotent group — the Malcev completion of any torsion free normal finite index subgroup of Γ — and $A(\Gamma)$ is the product $U(\Gamma) \times P(\Gamma)$ [5, 4.10, p. 93]. Thus Γ, acting on $U(\Gamma)$ by conjugation, is trivial on the commutator quotient $U(\Gamma)^{ab}$. Since $\varepsilon M_n(k)$ is abelian, this means that for any semi-simple $\rho \in R_n(\Gamma)$ we have

$$\mathrm{Hom}_\Gamma(\mathrm{Lie}(U(\Gamma)), \varepsilon M_n(k)) = \mathrm{Hom}_k(\mathrm{Lie}(U(\Gamma)), \varepsilon M_n(k)^\Gamma).$$

Now $M_n(k)^\Gamma$ is $\mathrm{End}_\Gamma(\rho)$ and, since ρ is semi-simple, this latter is a product of full matrix rings, whose sizes are equal to the multiplicities of the simple components of ρ. This identification of the subspace $\varepsilon M_n(k)^\Gamma$ of $\varepsilon M_n(k)$ in fact identifies it as a sub-Lie algebra of $\mathcal{G}\ell_n(k)$. From this we obtain our main result on singularities:

THEOREM 3.1. *Let Γ be nilpotent of rank at least two, and let $\rho \in R_n(\Gamma)$ be semi-simple. If some simple component of ρ has multiplicity two or more, then $CT_\rho(R_n(\Gamma))$ is not a linear subspace of $T_\rho(R_n(\Gamma))$. In particular, ρ is a singular point of $R_n(\Gamma)$.*

PROOF. The hypotheses imply that $\mathrm{End}_\Gamma(\rho)$ contains a subalgebra E isomorphic to a $n \times n$ matrix algebra, $n \geqq 2$. Inside E we can find elements A_1, A_2, B_1, B_2 such that $[A_1, A_2] = [B_1, B_2] = 0$ and the Lie subalgebra generated by $A_1 + B_1$, $A_2 + B_2$ non-nilpotent. (If E is 2×2, we could take $A_1 = e_{21}$, $A_2 = B_1 = 0$, and $B_2 = e_{12}$. Then the algebra generated by $A_1 + B_1$, $A_2 + B_2$ is $\mathrm{sl}_2(k)$.) Since $\mathrm{rank}(\Gamma) \geqq 2$, $\dim(U(\Gamma)^{ab}) \geqq 2$. Let $L = \mathrm{Lie}(U(\Gamma))$ and let $x_1, \ldots, x_n$ be a basis of L with $x_1, \ldots, x_d$ independent modulo $[L, L]$ and $x_{d+1}, \ldots, x_n$ a basis of $[L, L]$. Note that $d \geqq 2$. Define ϕ, ψ from l to $tM_n(k[[t]])$ by k-linearity and $\phi(x_1) = tA_1$, $\phi(x_2) = tA_2$, $\phi(x_i) = 0$ for $i > 2$, $\psi(x_1) = tB_1$, $\psi(x_2) = tB_2$ and $\psi(x_i) = 0$ for $i > 2$. Both ϕ and ψ are Lie homomorphisms, and in fact map $L/[L, L]$ to $tM_n(k[[t]])^\Gamma$, both of which are trivial Γ-modules, so ϕ and ψ are Γ-linear. Let $\bar{\phi}$, $\bar{\psi}$ denote their images under the map of (2.6). Suppose there were an f in $\mathrm{Hom}_\Gamma(L, tM_n(k[[t]]))$ whose image $\bar{f}$ in $\mathrm{Hom}_\Gamma(L, \varepsilon M_n(k))$ satisfied $\bar{f} = \bar{\phi} + \bar{\psi}$. Then $f(x_i) = t(A_i + B_i) +$ higher degree for $i = 1, 2$ and $f(x_i) = t \cdot 0 +$ higher degree for $i > 2$. By (2.6), this implies that the Lie subalgebra of $\mathcal{G}\ell_n(k)$ generated by $A_1 + B_1$ and $A_2 + B_2$ is a homomorphic image of $\mathrm{gr}(L)$, and hence nilpotent, contrary to the choice of A_i and B_i. It follows that there is no such f, and that the image of (2.5) is not a linear subspace, and then (2.3) gives our result.

It was shown in [4, 6.1, p. 93] that semi-simple representations of nilpotent groups that have no repeated components are non-singular points on the representation variety. This result can also be proved via the technique of curve tangents, using (2.4): for suppose $\rho:\Gamma\to \mathrm{GL}_n(k)$ is a semi-simple representation without repeated composition factors. Since Γ is nilpotent, $U(\Gamma)$ has trivial Γ-action. It follows that

$$(*)\qquad \mathrm{Hom}_\Gamma(U(\Gamma), I+\varepsilon M_n(k)) = \mathrm{Hom}_\Gamma(U(\Gamma)^{ab}, I+\varepsilon\,\mathrm{End}_\Gamma(\rho)).$$

Since $\mathrm{End}_\Gamma(\rho)$ is an abelian Lie algebra, reasoning as in (2.8) we have that (2.5) is surjective. It follows from (2.4) that ρ is non-singular on $R_n(\Gamma)$.

We can abstract this argument as follows: first, for any Γ and semi-simple ρ we have, by [4, 4.7, p. 73], that

$$(**)\qquad \mathrm{Hom}_\Gamma(U(\Gamma), I+\varepsilon M_n(k)) = H^1(\Gamma, \mathrm{Ad}\circ\rho).$$

Thus the condition needed to obtain (*) is that $H^1(\Gamma, (\mathrm{Ad}\circ\rho)/(\mathrm{Ad}\circ\rho)^\Gamma)=0$. Since $\mathrm{Ad}\circ\rho$ is semi-simple, for this latter it is suffcient that $H^1(\Gamma, M)=0$ for all simple Γ-modules other than k. This condition is used by Rudnick in [7, 1.1, p. 263], where it is called property P. (Rudnick notes further that nilpotent groups have property P [7, 1.7, p. 264].) Property P groups have non-singular varieties of irreducible representations [7, 2.3, p. 267]; in light of the above discussion this is also a corollary of (2.4).

PROPOSITION 3.2. *Assume that* $H^1(\Gamma, M)=0$ *for all non-trivial simple* Γ*-modules* (*property* P). *Let* $\rho:\Gamma\to \mathrm{GL}_n(k)$ *be a semi-simple representation without repeated composition factors. Then* ρ *is non-singular on* $\mathscr{R}_n(\Gamma)$ *and*

$$T_\rho(R_n(\Gamma)) = T_\rho(\mathscr{R}_n(\Gamma)) = Z^1(\Gamma, \mathrm{Ad}\circ\rho).$$

If in addition ρ *is simple, then* ρ *is non-singular on* $S_n(\Gamma)$ *and*

$$T_\rho(S_n(\Gamma)) = H^1(\Gamma, \mathrm{Ad}\circ\rho) = \mathrm{Hom}(\Gamma, k).$$

PROOF. We have already established all of (3.2) except for the final equality, which holds since k appears with multiplicity one in $\mathrm{Ad}\circ\rho$ for ρ simple, while its complement has trivial cohomology by property P.

As in [4, 6.2, p. 94] and [7, 2.3, p. 267], we can account for the tangents in $\mathrm{Hom}(\Gamma, k)$ from multiplication of ρ by linear characters.

4. Abelian-by-finite groups

This section concerns groups Γ with $U(\Gamma)$ finite-dimensional abelian, especially abelian-by-finite groups. For such groups, the results of Section 2 give us an explicit description of curve tangents and tangent cones. The spaces of simple representations of these groups were studied by Rudnick in [7]; this section relies heavily on that investigation.

We begin by summarizing the relevant results from Section 2:

PROPOSITION 4.1. *Assume $U(\Gamma)$ is finite-dimensional abelian and let $\rho \in R_n(\Gamma)$ be a semi-simple representation. Then*

$$CT_\rho(R_n(\Gamma)) = TC_\rho(R_n(\Gamma)) = \mathrm{Hom}_\Gamma(\mathrm{Lie}(U(\Gamma), \mathcal{G}\ell_n(k)) + B^1(\Gamma, \mathrm{Ad} \circ \rho).$$

If ρ is simple then also

$$CT_{[\rho]}(S_n(\Gamma)) = TC_{[\rho]}(S_n(\Gamma)) = \mathrm{Hom}_\Gamma(\mathrm{Lie}(U(\Gamma)), \mathcal{G}\ell_n(k)).$$

PROOF. Lie $(U(\Gamma))$ is graded nilpotent with support only in degree one. Then the equalities then follow from (2.9), and (2.8) and (2.3).

Note that (4.1) determines the dimension of $R_n(\Gamma)$ at ρ.

For the remainder of the section will fix notation: Γ *is an extension of the finite quotient group G by the torsion free abelian normal subgroup N.*

It is easy to see that $U(\Gamma) = U(N) = k \otimes N$, with this equality as both Γ and G-modules. To avoid confusion between Lie homomorphisms from $k \otimes N$ and module homomorphisms we always denote the set of the former by $\mathrm{Lie}_\Gamma(k \otimes N, -)$ or even $\mathrm{Lie}_\Gamma(N, -)$.

We note that if $\rho \in R_n(\Gamma)$ is a (semi-simple) representation and M the corresponding module then

$$\mathrm{Hom}_\Gamma(\mathrm{Lie}(U(\Gamma)), \mathcal{G}\ell_n(k)) = \mathrm{Lie}_G(N, \mathrm{End}_N(M)).$$

PROPOSITION 4.2. *Let $\rho \in R_n(\Gamma)$ be a semi-simple representation and let M be the corresponding module. Then:*

$$\mathrm{Lie}_G(N, \mathrm{End}_N(M)) + B^1(\Gamma, \mathrm{Ad} \circ \rho)$$
$$\subseteq T_\rho(R_n(\Gamma)) \subseteq \mathrm{Hom}_G(N, \mathrm{End}_N(M)) + B^1(\Gamma, \mathrm{Ad} \circ \rho).$$

If ρ is simple, then also

$$\mathrm{Lie}_G(N, \mathrm{End}_N(M)) \subseteq T_{[\rho]}(S_n(\Gamma)) \subseteq \mathrm{Hom}_G(N, \mathrm{End}_N(M)).$$

If ρ has no repeated N composition factors, then ρ is non-singular on $R_n(\Gamma)$ and reduced and non-singular on $\mathcal{R}_n(\Gamma)$. If ρ is simple, $[\rho]$ is non-singular on $S_n(\Gamma)$.

PROOF. The first assertions are the standard inclusions of (1.2a) taking the identities of (4.1) into account. If ρ has no repeated N composition factors $\mathrm{End}_N(M)$ is an abelian Lie algebra so Lie and module homomorphisms are equal and the inclusions (1.2a) become equalities. This establishes the non-singularity assertions.

We are now going to focus on simple representations, where the Clifford theory [1, 11.1, p. 259] comes into play. Let ρ be a simple representation of Γ. Then ρ is conjugate to a representation induced to Γ from a simple representation σ of a subgroup H containing N; $\sigma(N)$ is scalar and H is determined up to conjugacy [1, 11.1, p. 259]. Thus σ is a representation of the central-by-finite group $H/(H, N)$. Since central-by-finite groups are of type P [7, 1.7, p. 264], the closed subvariety $S_m(H/(H, N))$ of $S_m(H)$ is non-singular. If $m = n[\Gamma : H]^{-1}$, the representations $[\sigma]$ in $S_m(H)$ which induce to simple representations of Γ is open; intersection with $S_m(H/(H, N))$ gives a non-singular variety and the induction images of all these varieties cover $S_n(\Gamma)$. To say that a variety is covered by images of morphisms with non-singular domains is not especially helpful. We need to know also the nature of the maps on tangent spaces. The map in question is a composite, the first factor being the inclusion $S_m(H/(H, N)) \to S_m(H)$ and the second being the induction $S_m(H) \to S_n(\Gamma)$. The first is tangentially easy: it comes from the cohomology map $H^1(H/(H, N), \mathrm{Ad} \circ \rho) \to H^1(H, \mathrm{Ad} \circ \rho)$. The second is more complicated to explain infinitesimally. The following lemma, which also introduces Clifford theory notation, is necessary.

LEMMA 4.3. *Let ρ be a simple finite-dimensional representation of Γ with module M. Assume M has N-homogeneous components $V_1, \dots, V_r$, with $V_1 = L$. Let*

$$\Delta(\rho, N) = \Delta = \{\gamma \in \Gamma \mid \gamma L \text{ is isomorphic to } L\}$$

and let $g_1, \dots, g_r$ be coset representatives of Δ in Γ with $g_1 = e$. Then

$$\mathrm{End}_N(M) = k\Gamma \otimes_{k\Delta} \mathrm{End}_N(L),$$

PROOF. By Clifford theory, $M = k\Gamma \otimes_{k\Delta} L = L \oplus g_2 L \oplus \cdots \oplus g_r L$. Thus

$$\mathrm{End}_N(M) = \Pi\, \mathrm{End}_N(V_i) = \Pi g_i\, \mathrm{End}_N(L).$$

This shows that the right-hand side of the assertion of (4.3) maps onto the left. Since both sides have the same dimension, they are equal.

As a consequence of (4.3), we have the following determination of the effect of Clifford induction on tangent spaces:

PROPOSITION 4.4. *Let $\rho : \Gamma \to \mathrm{GL}_n(k)$ be simple and assume that ρ is conjugate by Clifford induction to $\mathrm{Ind}_\Delta^\Gamma(\sigma)$ for $\Delta = \Delta(\rho, N)$ and $\sigma : \Delta \to \mathrm{GL}_n(k)$ simple. Then we have an induced bijection on tangent cones:*

$$TC_{[\sigma]}(S_m(\Delta)) = CT_{[\sigma]} \to CT_{[\rho]} = TC_{[\rho]}(S_n(\Gamma)).$$

Moreover, $[\sigma]$ is scheme non-singular on $S_m(\Delta)$ if and only if $[\rho]$ is scheme non-singular on $S_n(\Gamma)$.

PROOF. Let L and M be the modules associated to σ and ρ respectively. Then by (4.1) and the remark following, we have

$$TC_{[\sigma]} = CT_{[\sigma]} = \mathrm{Lie}_\Delta(N, \mathrm{End}_N(L)) \quad \text{and} \quad TC_{[\rho]} = CT_{[\rho]} = \mathrm{Lie}_\Gamma(N, \mathrm{End}_N(M)).$$

By (4.3), and the universal property of induction, we have

$$\mathrm{Hom}_\Gamma(N, \mathrm{End}_N(M)) = \mathrm{Hom}_\Delta(N, \mathrm{End}_N(L)),$$

the identity given by projection onto $\mathrm{End}_N(L)$ as a Δ-factor of $\mathrm{End}_N(M)$. Since it is clear that this identity preserves Lie homomorphisms, we obtain the first assertion of (4.4). By (4.2), scheme non-singularity is equivalent to all homomorphisms being Lie homomorphisms, and the above identity shows that this is the same assertion for both $[\sigma]$ and $[\rho]$.

Clifford theory is most helpful when Δ is a proper subgroup of Γ, i.e., when N is not represented centrally.

Since we have calculated tangent cones in (4.1) and (4.4), we have calculated dimensions. We record this as follows:

COROLLARY 4.5. *Let $[\rho] \in S_n(\Gamma)$, let $\Delta = \Delta(N, \rho)$ and assume $[\rho]$ is induced from $[\sigma]$ in $S_m(\Delta)$. Then the dimension of $S_n(\Gamma)$ at ρ equals* $\dim(\mathrm{Lie}_\Delta(N, \mathrm{End}_N(\sigma)))$.

We apply this to some examples.

EXAMPLE 4.6. Let $\Gamma = N \rtimes S_3$ where the symmetric group S_3 acts on

$$N = \{(x, y, z) \in \mathbf{Z}^{(3)} \mid x + y + z = 0\}$$

by permutation (see [4, p. 42]). Let $\rho \in R_2(\Gamma)$ be the simple representation preceded by projection on S_3 and let V be its module. As shown in [4, p. 43], $\mathrm{End}_N(Y) = V_1 \oplus V_2 \oplus V$ where V_1 is the trivial and V_2 the non-trivial character of S_3. It follows that $\mathrm{Hom}_\Gamma(N, \mathrm{End}_N(V))$ is one-dimensional, and it is clear that any two-dimensional abelian subalgebra of $\mathrm{End}_k(V)$ must meet $kI = V_1$. So $\mathrm{Lie}_\Gamma(N, \mathrm{End}_N(V)) = 0$. Thus $CT_{[\rho]} = TC_{[\rho]}(S_2(\Gamma)) = \{0\}$. It follows that $[\rho]$ is an isolated point of $S_2(\Gamma)$, with

$$H^1(\Gamma, \mathrm{Ad} \circ \rho) = \mathrm{Hom}_\Gamma(N, \mathrm{End}_k(V)) \neq 0.$$

In fact, the same argument shows that every point in $S_2(\Gamma)$ is isolated. These results are obtained by calculations in [4, pp. 42–43].

EXAMPLE 4.7. Let $\Gamma = N \rtimes D_4$, where D_4 is the dihedral group of order 8 and $N = \mathbf{Z}[D_4]$ (integral group ring). Let $\rho \in R_2(\Gamma)$ be the simple representation of D_4 preceded by projection on D_4. From the character table of D_4 [1, p. 220] it is clear that $\mathrm{End}_N(V)$ (where V is the module of ρ) is the sum of the four linear characters of D_4. Thus $\mathrm{Lie}_\Gamma(N, \mathrm{End}_k(V))$ is the union of the three planes $\mathrm{Hom}_\Gamma(k, k) + \mathrm{Hom}_\Gamma(V_i, V_i)$ where V_i, $i = 1, 2, 3$, ranges over the three non-trivial one dimensional D_4-modules. Thus $CT_{[\rho]} = TC_{[\rho]}(S_2(\Gamma))$ is the union of three planes. We can account for these as follows: there are three index two subgroups H_i, $i = 1, 2, 3$ of Γ containing N. Induction from H_i of one-dimensional modules yields a locally closed subset whose closure C_i is two-dimensional (since $\mathrm{Hom}_{H_i}(N, k)$ is two dimensional) and these three sets meet at $[\rho]$, yielding the three planes in $TC_{[\rho]}$ as tangents.

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